Counting smaller trees in the Tamari order

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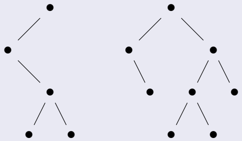
April 5, 2013

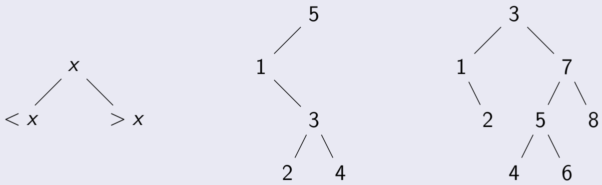
1. Introduction
   - Basic definitions
   - Tamari lattice
   - Goal
2. Our work
   - Main result
   - Example

Binary trees

Binary trees

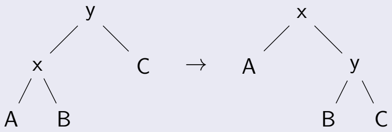


Canonical labelling



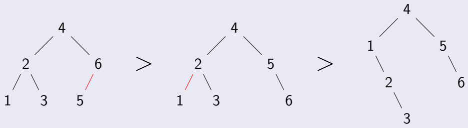
Order relation

Right rotation



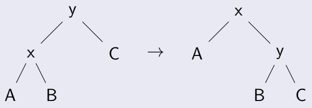
This gives an order relation on binary trees.

Example



The Tamari lattice

Right rotation



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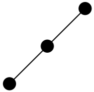

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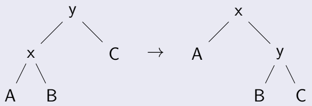

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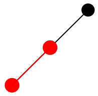

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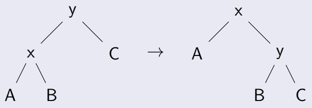

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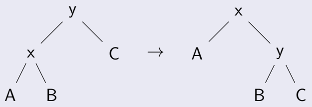

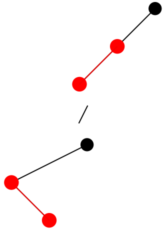

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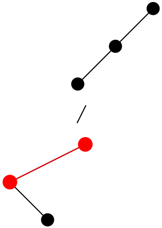
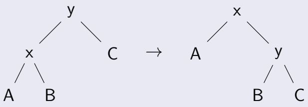
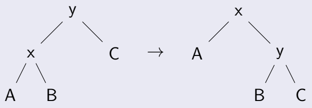


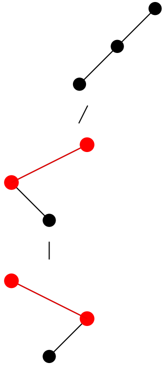
Right rotation



The Tamari lattice

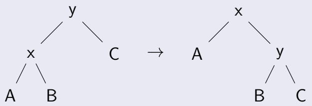
Right rotation

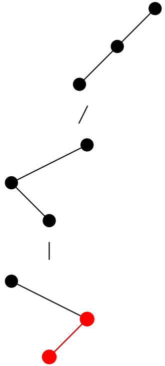




The Tamari lattice

Right rotation





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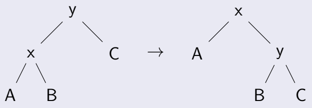

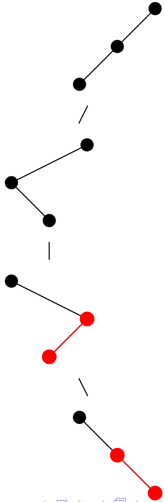

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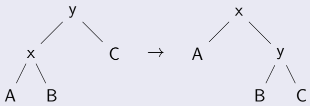

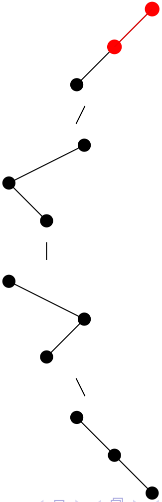

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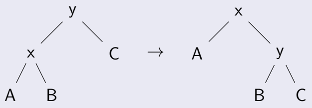

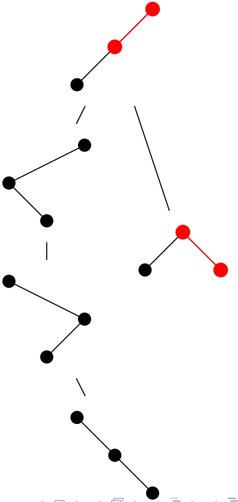

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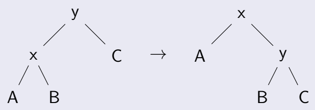

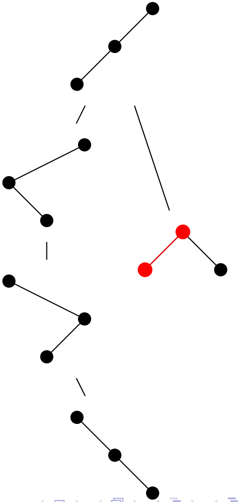

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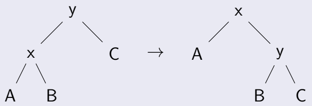

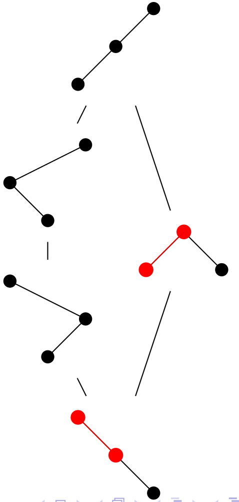

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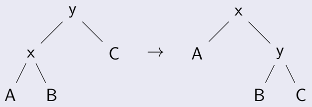

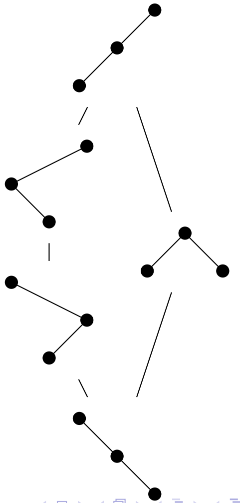

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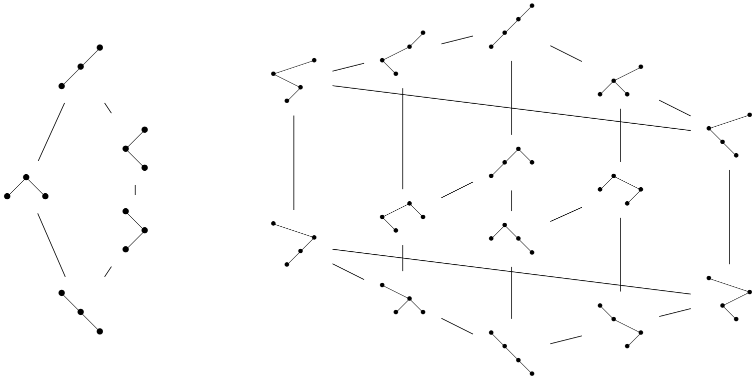


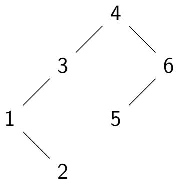
Figure: The Tamari lattices of size 3 and 4.

Our objective

Goal

We want a formula that computes, for any given tree T the number of trees smaller than T in the Tamari order.

Example : how many trees are smaller than or equal to this one ?



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Tamari polynomials

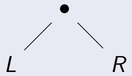
Tamari polynomials

Given a binary tree T, we define:

$$\mathcal{B}_{\emptyset} := 1 \qquad (1)$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x-1} \qquad (2)$$

with T =



Theorem

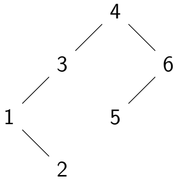
Theorem

Let T be a binary tree. Its Tamari polynomial $\mathcal{B}_T(x)$ counts the trees smaller than or equal to T in the Tamari order according to the number of nodes on their left border. In particular, $\mathcal{B}_T(1)$ is the number of trees smaller than T.

Example

$$\mathcal{B}_\emptyset := 1$$

$$\mathcal{B}_T(x) := x\mathcal{B}_L(x)\frac{x\mathcal{B}_R(x) - \mathcal{B}_R(1)}{x - 1}$$



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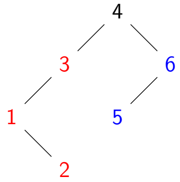

- $\mathcal{B}_T(x) = \mathcal{B}_4(x) = x\mathcal{B}_3(x)\frac{\mathcal{B}_6(x)-\mathcal{B}_6(1)}{x-1}$

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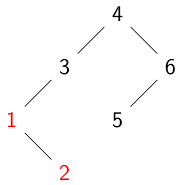

- $\mathcal{B}_T(x) = \mathcal{B}_4(x) = x\mathcal{B}_3(x)\frac{\mathcal{B}_6(x)-\mathcal{B}_6(1)}{x-1}$
- $\mathcal{B}_3(x) = x\mathcal{B}_1(x)$

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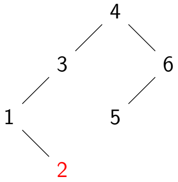

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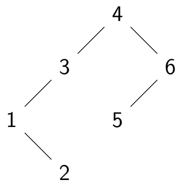

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- $\mathcal{B}_2(x) = x$

Example

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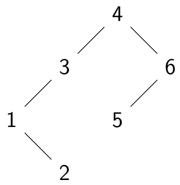

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- $\mathcal{B}_2(x) = x$
- $\mathcal{B}_1(x) = x(1+x) = x + x^2$

Example

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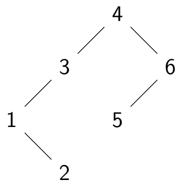

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- $\mathcal{B}_3(x) = x(x+x^2) = x^2 + x^3$

Example

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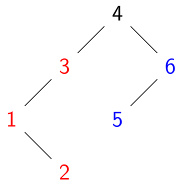

- $\mathcal{B}_T(x) = \mathcal{B}_4(x) = x(x^2 + x^3)\frac{\mathcal{B}_6(x) - \mathcal{B}_6(1)}{x-1}$

Example

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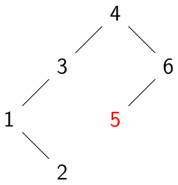

- $\mathcal{B}_T(x) = \mathcal{B}_4(x) = x(x^2 + x^3)\frac{\mathcal{B}_6(x)-\mathcal{B}_6(1)}{x-1}$
- $\mathcal{B}_6(x) = x\mathcal{B}_5(x)$

Example

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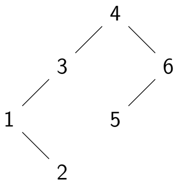

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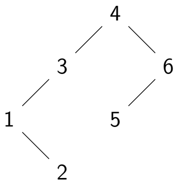

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- $\mathcal{B}_6(x) = x\mathcal{B}_5(x)$
- $\mathcal{B}_5(x) = x$
- $\mathcal{B}_6(x) = x \cdot x = x^2$

Example

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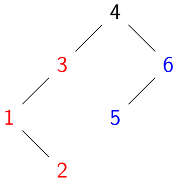

- $\mathcal{B}_4(x) = x(x^2 + x^3)(1 + x + x^2)$

Example

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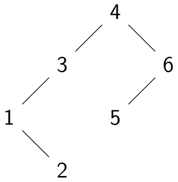
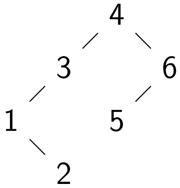
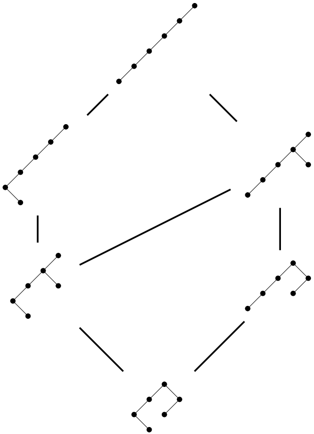
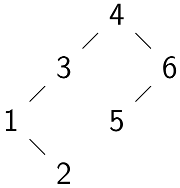


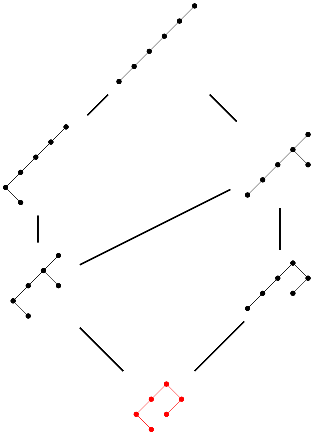
- $\mathcal{B}_4(x) = x(x^2 + x^3)(1 + x + x^2)$
- $\mathcal{B}_4(x) = x^6 + 2x^5 + 2x^4 + x^3$





$$\mathcal{B}_T(x) = x^3 + 2x^4 + 2x^5 + x^6$$
$$\mathcal{B}_T(1) = 6$$





$$\mathcal{B}_T(x) = {\color{red}x^3} + 2x^4 + 2x^5 + x^6$$
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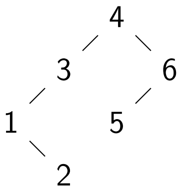

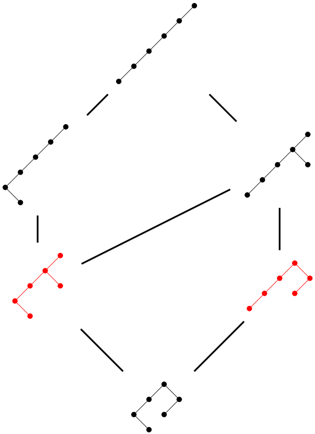

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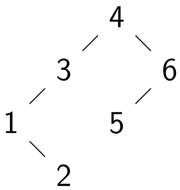

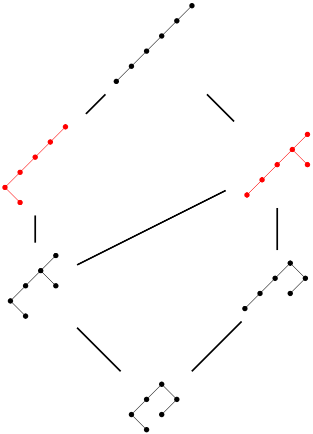

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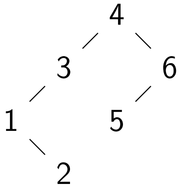
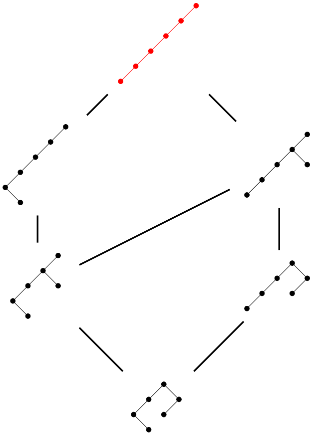
$$\mathcal{B}_T(1) = 6$$





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